Discrete-time Constant-size Model with Seed Dormancy

Differential Equations

Matching-alleles model with Seed Dormancy. Life-cycle diagram is shown in figure S2.

```
βSub = {β[1, 1] → X, β[1, 2] → Y, β[2, 1] → Y, β[2, 2] → X};
```

$$\text{WH2}[i\_, t\_] := 1 - \alpha H\,\text{Sum}\left[\beta[i, j]\,(1 - pP[t])^{j-1}\,pP[t]^{\text{Mod}[j,2]}, \{j, 1, 2\}\right] \,/.\, \beta\text{Sub}$$
$$\text{WP2}[j\_, t\_] := 1 + \alpha P\,\text{Sum}\left[\beta[i, j]\,(1 - pH[t])^{i-1}\,pH[t]^{\text{Mod}[i,2]}, \{i, 1, 2\}\right] \,/.\, \beta\text{Sub}$$

$$\text{WHbar2}[t\_] := pH[t] \times \text{WH2}[1, t] + (1 - pH[t])\,\text{WH2}[2, t]$$
$$\text{WPbar2}[t\_] := pP[t] \times \text{WP2}[1, t] + (1 - pP[t])\,\text{WP2}[2, t]$$

$\frac{dpJ}{dt} = j1[t]$, $\frac{dpH}{dt} = f1[t]$, $\frac{dpP}{dt} = f2[t]$.

$$j1[t\_] := (1 - s)\,pJ[t] + s\,pH[t]\,\frac{\text{WH2}[1, t]}{\text{WHbar2}[t]};$$
$$f1[t\_] := (1 - s)\,pH[t]\,\frac{\text{WH2}[1, t]}{\text{WHbar2}[t]} + s\,pJ[t];$$
$$f2[t\_] := pP[t]\,\frac{\text{WP2}[1, t]}{\text{WPbar2}[t]}$$

Equilibria

```
Solve[{j1[t] == pJ[t], f1[t] == pH[t], f2[t] == pP[t]}, {pJ[t], pH[t], pP[t]}]
```

$$\left\{\{pJ[t] \to 0, pH[t] \to 0, pP[t] \to 0\},\ \{pJ[t] \to 0, pH[t] \to 0, pP[t] \to 1\},\ \left\{pJ[t] \to \frac{1}{2}, pH[t] \to \frac{1}{2}, pP[t] \to \frac{1}{2}\right\},\ \{pJ[t] \to 1, pH[t] \to 1, pP[t] \to 0\},\ \{pJ[t] \to 1, pH[t] \to 1, pP[t] \to 1\}\right\}$$

Stability

```
jMtrx = {{D[j1[t], pJ[t]], D[j1[t], pH[t]], D[j1[t], pP[t]]},
    {D[f1[t], pJ[t]], D[f1[t], pH[t]], D[f1[t], pP[t]]},
    {D[f2[t], pJ[t]], D[f2[t], pH[t]], D[f2[t], pP[t]]}} /.
  {pH[t] → 1/2, pH[t] → 1/2, pP[t] → 1/2} // Simplify
```

$$\left\{\left\{1 - s, s, \frac{s\,(X - Y)\,\alpha H}{-2 + X\,\alpha H + Y\,\alpha H}\right\}, \left\{s, 1 - s, -\frac{(-1 + s)\,(X - Y)\,\alpha H}{-2 + X\,\alpha H + Y\,\alpha H}\right\}, \left\{0, \frac{(X - Y)\,\alpha P}{2 + X\,\alpha P + Y\,\alpha P}, 1\right\}\right\}$$

```
jMtrx // MatrixForm
```

$$\begin{pmatrix} 1-s & s & \frac{s\,(X-Y)\,\alpha H}{-2+X\,\alpha H+Y\,\alpha H} \\ s & 1-s & -\frac{(-1+s)\,(X-Y)\,\alpha H}{-2+X\,\alpha H+Y\,\alpha H} \\ 0 & \frac{(X-Y)\,\alpha P}{2+X\,\alpha P+Y\,\alpha P} & 1 \end{pmatrix}$$

```
charpoly = Det[λ IdentityMatrix[3] - jMtrx];
```

We make the simplifying assumption that αH=αP=α.

We can test stability by using the Routh-Hurwitz equations on a transformed characteristic polynomial.

```
Clear[f, a, b, c, d]
f[x_] := x^3 + b x^2 + c x + d
Collect[Factor[Collect[(z - 1)^3 (f[(z + 1)/(z - 1)]), z]], z]
```

$$1-b+c-d+(3-b-c+3\,d)\,z+(3+b-c-3\,d)\,z^2+(1+b+c+d)\,z^3$$

```
temp = Collect[Numerator[charpoly /. {αH → α, αP → α}
      (*/.Flatten[Solve[{a==α X,Y==y X},{α,Y}]]*)] /
    Coefficient[Numerator[charpoly /. {αH → α, αP → α}
      (*/.Flatten[Solve[{a==α X,Y==y X},{α,Y}]]*)], λ^3], λ, Simplify];
{d, c, b} = CoefficientList[temp, λ][[1 ;; 3]];
{a0, a1, a2, a3} =
 CoefficientList[Collect[Factor[Expand[(z - 1)^3 (f[(z + 1)/(z - 1)])]], z], z] // Simplify
```

$$\left\{\frac{-32+6\,X^2\,\alpha^2+20\,X\,Y\,\alpha^2+6\,Y^2\,\alpha^2-s\,(-32+5\,X^2\,\alpha^2+22\,X\,Y\,\alpha^2+5\,Y^2\,\alpha^2)}{(-2+X\,\alpha+Y\,\alpha)\,(2+X\,\alpha+Y\,\alpha)},\right.$$
$$\frac{4\,(X-Y)^2\,\alpha^2+s\,(-32+X^2\,\alpha^2+30\,X\,Y\,\alpha^2+Y^2\,\alpha^2)}{(-2+X\,\alpha+Y\,\alpha)\,(2+X\,\alpha+Y\,\alpha)},$$
$$\left.\frac{(-2+5\,s)\,(X-Y)^2\,\alpha^2}{(-2+X\,\alpha+Y\,\alpha)\,(2+X\,\alpha+Y\,\alpha)},\ -\frac{s\,(X-Y)^2\,\alpha^2}{(-2+X\,\alpha+Y\,\alpha)\,(2+X\,\alpha+Y\,\alpha)}\right\}$$

```
Reduce[{a0 > 0, a1 > 0, a2 > 0, a3 > 0, FullSimplify[a2 a1 - a3 a0] > 0
   (*,0<y<1,0<a<1*), 0 < Y < X < 1, 0 < s < 1}, s] // FullSimplify
```

$$Y > 0 \,\&\&\, \left(0 < \alpha < \frac{2}{\sqrt{9X^2 - 14XY + 9Y^2}} \;||\; -\frac{2}{\sqrt{9X^2 - 14XY + 9Y^2}} < \alpha < 0\right) \&\&$$

$$Y < X < 1 \,\&\&\, \frac{1}{32}\left(-\frac{4 + (X - 3Y)(3X - Y)\alpha^2}{-1 + XY\alpha^2} - \sqrt{\frac{(-2 + (X + Y)\alpha)(2 + (X + Y)\alpha)(-4 + (9X^2 - 14XY + 9Y^2)\alpha^2)}{(-1 + XY\alpha^2)^2}}\right) < s < \frac{1}{32}\left(-\frac{4 + (X - 3Y)(3X - Y)\alpha^2}{-1 + XY\alpha^2} + \sqrt{\frac{(-2 + (X + Y)\alpha)(2 + (X + Y)\alpha)(-4 + (9X^2 - 14XY + 9Y^2)\alpha^2)}{(-1 + XY\alpha^2)^2}}\right)$$

$$\text{critUpper[X\_, }\alpha\text{\_] := } \frac{1}{32}\left(-\frac{4 + (X - 3Y)(3X - Y)\alpha^2}{-1 + XY\alpha^2} + \sqrt{\frac{(-2 + (X + Y)\alpha)(2 + (X + Y)\alpha)(-4 + (9X^2 - 14XY + 9Y^2)\alpha^2)}{(-1 + XY\alpha^2)^2}}\right)$$

$$\text{critLower[X\_, }\alpha\text{\_] := } \frac{1}{32}\left(-\frac{4 + (X - 3Y)(3X - Y)\alpha^2}{-1 + XY\alpha^2} - \sqrt{\frac{(-2 + (X + Y)\alpha)(2 + (X + Y)\alpha)(-4 + (9X^2 - 14XY + 9Y^2)\alpha^2)}{(-1 + XY\alpha^2)^2}}\right)$$

```
λnum[αin_, sin_] := Sort[
   Abs[Eigenvalues[jMtrx /. {αH → α, αP → α, X → 1, Y → 0} /. {s → sin, α → αin}]]][[-1]]
```

Checking analytical conditions with numerical test of stability

```
Show[RegionPlot[λnum[αin, sin] < 1, {αin, 0, 0.5}, {sin, 0, 0.5}],
 Plot[{critUpper[X, α] /. {X → 1, Y → 0}, critLower[X, α] /. {X → 1, Y → 0}},
  {α, 0, 0.5}, PlotStyle → Red]]
```

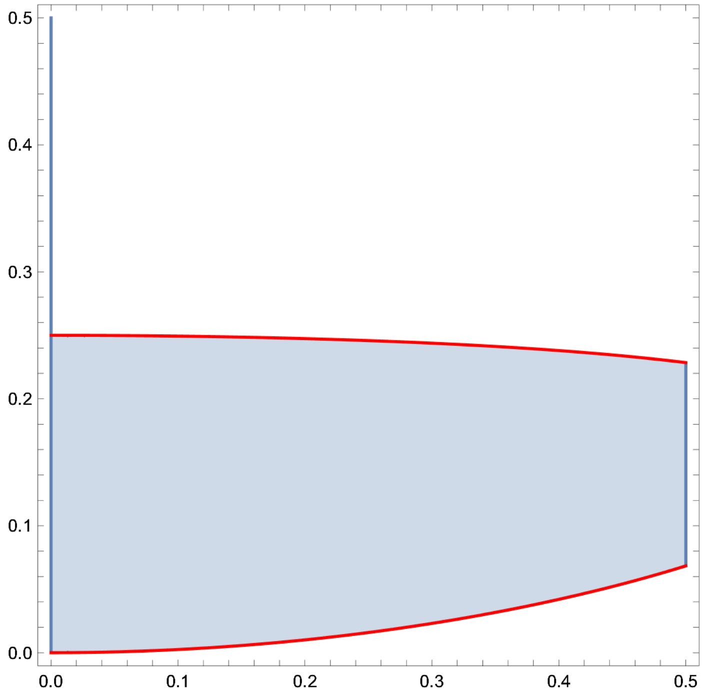

Figure S2

```
Plot[{critUpper[X, α] /. {X → 1, Y → 0}, critLower[X, α] /. {X → 1, Y → 0}}, {α, 0, 0.5},
 PlotStyle → Black, Filling → 1 → {2}, FillingStyle → Directive[Gray], Frame → True,
 FrameTicks → {{True, False}, {True, False}}, FrameTicksStyle → Directive[Black, 14]]
Export[NotebookDirectory[] <> "Plots/SeedDormancy.pdf", %, ImageResolution → 300];
```